Vector and Spinor Decomposition of SU(2) Gauge Potential, their Equivalence and Knot Structure in SU(2) Chern-Simons Theory

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Abstract

In this paper, spinor and vector decomposition of SU(2) gauge potential are presented and their equivalence is constructed using a simply proposal. We also obtain the action of Faddeev nonlinear $O(3)$ sigma model from the SU(2) massive gauge field theory which is proposed according to the gauge invariant principle. At last, the knot structure in SU(2) Chern-Simons filed theory is discussed in terms of the ϕ–mapping topological current theory. The topological charge of the knot is characterized by the Hopf indices and the Brouwer degrees of ϕ-mapping.

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I.   INTRODUCTION

Since the decomposition theory of gauge potential reveals the inner structure of gauge potential, it inputs the geometrical and topological information to the gauge potential (i.e. the connection of principal bundle), and establishes a direct relationship between differential geometry and topology of gauge field. The above viewpoint of inner-structure is expected to enrich the gauge theory with deeper physical contents; actually an outstanding case in point is the general relativity. In general relativity or 4-dimensional Riemannian geometry, the connection can be expressed in terms of the fundamental field or metric $g_{\mu\nu}$ which is necessary to be introduced to describe the gravity field. In recent years the decomposition theory of gauge potential has played a more and more important role in theoretical physics and mathematics. From this viewpoint much progress has been made by other authors[1, 2] and by us, such as the decomposition of U(1) gauge potential and U(1) Chern-Simons, the decomposition of SU(2) connection and the Skyrme theory, the decomposition of SU(N) connection and the effective theory of SU(N) QCD, the decomposition of SO(N) spin connection and the structure of GBC topological current [1, 2, 3, 4, 5, 6, 7].

Cho[7] derive a generalized Skyrme action from the Yang-Mills action of SU(2) QCD, which is proposed to be an effective action of SU(2) QCD in the infrared limit. This stimulate us to explore the relationship between SU(2) Yang-Mills theory and the Faddeev nonlinear $O(3)$ sigma model which has intriguing consequences[8]. One of the co-authors Prof.Duan[9] have pointed out almost twenty years ago that gauge potential A_μ can be decomposed into two parts: a_μ and b_μ. Here a_μ satisfies the gauge transformation $a'_\mu = ga_\mu g^{-1} + \partial_\mu gg^{-1}$, and the b_μ satisfies the adjoint transformation $b'_\mu = gb_\mu g$. The a_μ part may show the geometry property of system and the b_μ part may be looked upon as vector boson which would be massive without introducing Higgs mechanism and spontaneous symmetry breaking. From this point of view, we can introduce a massive term in terms of b_μ part to construct a massive gauge field theory without destroying the gauge invariance which can naturally deduce the action of Faddeev nonlinear $O(3)$ sigma model using the vector decomposition of SU(2) gauge potential.

In this paper, spinor and vector decomposition of SU(2) gauge potential are presented and their equivalence is constructed using a simply proposal. We also obtain the action of Faddeev nonlinear $O(3)$ sigma model from the SU(2) massive gauge field theory which is

proposed according to the gauge invariant principle. At last, the knot structure in SU(2) Chern-Simons filed theory is discussed in terms of the ϕ–mapping topological current theory proposed by Prof.Duan. The topological charge of the knot is characterized by the Hopf indices and the Brouwer degrees of ϕ-mapping.

II.   DECOMPOSITION OF SU(2) CONNECTION

We begin with a brief review of our previous work[6] on the spinor decomposition of SU(2) connection. Let M be a compact oriented 4-dimensional manifold, on which the principal bundle $P(\pi, M, SU(2))$ is defined. It is well known that in the SU(2) gauge field theory with spinor representation Ψ , the covariant derivative is defined as

$$D_\mu \Psi = \partial_\mu \Psi - \frac{1}{2i} A_\mu^a \sigma^a \Psi, \tag{1}$$

where

$$A = A_\mu dx^\mu = \frac{1}{2i} A_\mu^a \sigma^a dx^\mu \tag{2}$$

is the SU(2) gauge potential, i.e. the connection of principle bundle P, and $T^a = \frac{1}{2i}\sigma^a (a = 1, 2.3)$ are the SU(2) generator with σ^a being Pauli matrix. The final result of spinor decomposition is

$$\begin{aligned} A_\mu^a &= \frac{i}{\Psi^\dagger \Psi}(\Psi^\dagger \sigma^a \partial_\mu \Psi - \partial_\mu \Psi^\dagger \sigma^a \Psi) \\ &\quad - \frac{i}{\Psi^\dagger \Psi}(\Psi^\dagger \sigma^a D_\mu \Psi - D_\mu \Psi^\dagger \sigma^a \Psi). \end{aligned} \tag{3}$$

The traditional decomposition theory of gauge potential always uses the parallel field condition $D_\mu \Psi = 0$ and the normalized spinor Ψ with $\Psi^\dagger \Psi = 1$, so the vectorial transformation part of A_μ^a disappears and A_μ^a can be expressed as

$$A_\mu^a = i(\Psi^\dagger \sigma^a \partial_\mu \Psi - \partial_\mu \Psi^\dagger \sigma^a \Psi), \tag{4}$$

which satisfies the SU(2) gauge transformation. The expression (4) is useful to reveal the inner structure of the second Chern class. Noticing the index a in gauge potential A_μ^a, the gauge potential A_μ^a can also be described as the vector form $\vec{A}_\mu$.

Another method of decomposing the SU(2) connection has been done by Cho[2] and Faddeev[1] in terms of the vector topological field $\vec{n}$ which is taken as the basic field on the

manifold M. The covariant derivative of $\vec{n}$ is defined as

$$D_\mu \vec{n} = \partial_\mu \vec{n} - \vec{A}_\mu \times \vec{n}. \tag{5}$$

The decomposition formula of $\vec{A}_\mu$ is

$$\vec{A}_\mu = C_\mu \vec{n} + \vec{n} \times \partial_\mu \vec{n} + \vec{X}_\mu, \tag{6}$$

where $C_\mu = \vec{A}_\mu \cdot \vec{n}$ is the projection of gauge potential $\vec{A}_\mu$ on $\vec{n}$ direction and $\vec{X}_\mu = \vec{n} \times D_\mu \vec{n}$ which is perpendicular with $\vec{n}$. Observed that $\vec{n}$ represents a direction which is purely associated with orientation of the moving frame $\{\vec{n}, \partial_\mu \vec{n}, \vec{n} \times \partial_\mu \vec{n}\}$, one can express $\vec{X}_\mu$ in terms of the moving frame as

$$\vec{X}_\mu = f_1 \partial_\mu \vec{n} + f_2 \vec{n} \times \partial_\mu \vec{n}. \tag{7}$$

The decomposition of SU(2) connection (3) and (6) should be equivalent. The topological field $\vec{n}$ which selects the color direction at each space-time point[7] can be constructed using the spinor field like this

$$n^a = \Psi^\dagger \sigma^a \Psi. \tag{8}$$

Once the relationship between the topological filed $\vec{n}$ and the spinor field Ψ is set up, the projection of gauge potential C_μ can be calculated where

$$C_\mu = A_\mu^a n^a = 2i\Psi^\dagger \partial_\mu \Psi, \tag{9}$$

and the decomposition formula (4) can be directly driven from (6) under the gauge parallel condition $D_\mu \vec{n} = 0$ after a simple calculation. Then, two kinds of the decomposition of the SU(2) connection is proved to be equivalent.

One of the co-authors Prof.Duan have pointed out almost twenty years ago that gauge potential should be decomposed in terms of the gauge covariant $A_\mu = a_\mu + b_\mu$, which the a_μ satisfies the gauge transformation $a'_\mu = ga_\mu g^{-1} + \partial_\mu gg^{-1}$, and the b_μ satisfies the adjoint transformation $b'_\mu = gb_\mu g$. The a_μ part may show the geometry property of system and the b_μ part may be looked upon as vector boson which would be massive.

Under the infinitesimal gauge transformation[2]

$$\delta \vec{n} = -\vec{\alpha} \times \vec{n}, \quad \delta \vec{A}_\mu = D_\mu \vec{\alpha}, \tag{10}$$

one have

$$\delta C_\mu = \vec{n} \cdot \partial_\mu \vec{\alpha}, \ \delta \vec{X}_\mu = -\vec{\alpha} \times \vec{X}_\mu. \tag{11}$$

$\vec{X}_\mu$ part in decomposition formula (6) transforms covariantly under the gauge transformation. Then we can introduce SU(2) massive gauge field theory which the Lagrange is defined like this without destroying the gauge covariant

$$\mathcal{L} = -\frac{1}{4} \vec{F}_{\mu\nu} \cdot \vec{F}_{\mu\nu} + \lambda \vec{X}_\mu^2. \tag{12}$$

One can simply calculate the massive term in the above Lagrange from (7) which gives

$$\lambda \vec{X}_\mu^2 = \lambda (f_1^2 + f_2^2) \partial_\mu \vec{n}^2. \tag{13}$$

Noticing that this massive term is just the first term in the Faddeev's model which the action is defined as

$$S = \int d^4x \{ m^2 (\partial_\mu \vec{n})^2 + \frac{1}{e^2} (\vec{n} \cdot \partial_\mu \vec{n} \times \partial_\nu \vec{n})^2 \}, \tag{14}$$

and the second term named Faddeev-Skyrme term is so familiar that we are not to discuss in detail, so one can really deduce the action of Faddeev's model which is a unique action for describing SU(2) Yang-Mills theory at low energies from the Lagrange of SU(2) massive gauge field. The massive gauge field theory constructed here may be a suitable theory for describing the interaction of elementary particle. A further discussion of the quantization and renormalization of massive gauge filed theory will be published elsewhere.

III. KNOT STRUCTURE IN SU(2) CHERN-SIMONS ACTION

Chern-Simons action is a very important topological invariant which have deep relationship with the knot invariant as pointed out by E.Witten in his pioneer work[10]. In fact, Duan has pointed out in [3] that U(1) Chern-Simons action is an important invariant required to describe the topology of knot in Chern-Simons field theory. Here, we will study the knot structure in SU(2) Chern-Simons field theory.

Chern-Simons action is the integral of the Chern-Simons $3-$form[11]

$$S = \frac{1}{8\pi^2} \int_V Tr(A \wedge dA - \frac{2}{3} A \wedge A \wedge A), \tag{15}$$

where V is the space volume. It can also be expressed as

$$S = -\frac{1}{16\pi^2} \int_V \epsilon^{\mu\nu\lambda} (\vec{A}_\mu \cdot \partial_\nu \vec{A}_\lambda - \frac{1}{3} \vec{A}_\mu \cdot \vec{A}_\nu \times \vec{A}_\lambda) d^3x. \tag{16}$$

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Using the decomposition formula (6) and the condition $D_\mu \vec{n} = 0$, one can easily drive at

$$S = -\frac{1}{16\pi^2} \int_V (\epsilon^{\mu\nu\lambda} C_\mu \partial_\nu C_\lambda + \epsilon^{\mu\nu\lambda} C_\mu H_{\nu\lambda}) d^3x, \tag{17}$$

where $H_{\mu\nu} = \vec{n} \cdot \partial_\mu \vec{n} \times \partial_\nu \vec{n}$ plays an essential role in studying the knot structure in SU(2) Chern-Simons field theory. The normalized two component spinor Ψ can be expressed by

$$\Psi = \begin{pmatrix} l^0 + il^1 \\ l^2 + il^3 \end{pmatrix}, \tag{18}$$

where $l^a (a = 0, 1, 2, 3)$ is a real unit vector. After some algebra, the first term in Eq.(17) is

$$S^{(1)} = \frac{1}{12\pi^2} \int \epsilon_{abcd} \epsilon^{\mu\nu\lambda} l^a \partial_\mu l^b \partial_\nu l^c \partial_\lambda l^d d^3x, \tag{19}$$

which is just the winding number of Gauss mapping $S^3 \mapsto S^3$. Under SU(2) gauge transformation, the Chern-Simons action transforms like this

$$S' = S + m, \tag{20}$$

where

$$\omega = \int_V Tr(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg) \tag{21}$$

is the winding number of map $S^3 \mapsto SU(2)$ with g being an element of SU(2). To see more clearer, g can be parameterized as

$$g = l^a s^a, \quad s^a = (I, i\vec{\sigma}). \tag{22}$$

Then one exactly get Eq(19) from (21). So the firse term in Chern-Simons action (17) can be wiped off after gauge transformation. Now we focus on the second term of Eq.(17)

$$S = \frac{1}{16\pi^2} \int_V \epsilon^{\mu\nu\lambda} C_\mu H_{\nu\lambda} d^3x. \tag{23}$$

We derive the 2-dimensional topological current from the field tensor $H_{\mu\nu}$, and show that there are knot structures inhering in SU(2) Chern-Simons. Int fact, the unit vector $\vec{n}$ is the section of sphere bundle S^2. Defining a 2-component vector $\vec{\phi} = (\phi^1, \phi^2)$ on this S^2, i.e. $\tilde{\phi} \cdot \vec{n} = 0(\tilde{\phi} = \phi^a/\|\phi\|, a = 1, 2)$, it can be proved that[12]

$$H_{\mu\nu} = 2\epsilon_{ab} \partial_\mu \tilde{\phi}^a \partial_\nu \tilde{\phi}^b. \tag{24}$$

According to ϕ-mapping topological current theory[13], the 2-dimensional topological current is defined as

$$j^\lambda = \frac{1}{4\pi}\epsilon^{\lambda\mu\nu}\epsilon_{ab}\partial_\mu\tilde{\phi}^a\partial_\nu\tilde{\phi}^b, \tag{25}$$

so we have

$$j^\lambda = \frac{1}{8\pi}\epsilon^{\lambda\mu\nu}H_{\mu\nu}. \tag{26}$$

Then using $\partial_\mu\tilde{\phi}^a = \partial\phi^a/\|\phi\| + \phi^a/\partial_\mu(1/\|\phi\|)$ and the Green function formula in $\phi-$space $\partial_a\partial_a ln\|\phi\| = 2\pi\delta^2(\vec{\phi})$ with $\partial_a = \partial/\partial\phi^a$, it can be proved that

$$j^\lambda = \delta^2(\phi)D^\lambda(\frac{\phi}{x}), \tag{27}$$

where

$$D^\lambda(\frac{\phi}{x}) = \frac{1}{2}\epsilon^{\lambda\mu\nu}\epsilon_{ab}\partial_\mu\phi^a\partial_\nu\phi^b \tag{28}$$

is the Jacobian vector. This expression of j^λ provides an important conclusion

$$j^\lambda \begin{cases} = 0 & \text{if and only if } \vec{\phi} \neq 0, \\ \neq 0 & \text{if and only if } \vec{\phi} = 0. \end{cases} \tag{29}$$

So it is necessary to study the zero points of $\vec{\phi}$ to determine the nonzero solution of j^λ. The implicit function theory[14] show that under the regular condition

$$D^\lambda(\frac{\phi}{x}) \neq 0, \tag{30}$$

the general solutions of

$$\phi^1(x^1, x^2, x^3) = 0, \phi^2(x^1, x^2, x^3) = 0, \tag{31}$$

can be expressed as

$$x^1 = x_k^1(s), x^2 = x_k^2(s), x^3 = x_k^3(s). \tag{32}$$

which represent N isolated singular strings $L_k(k = 1, 2, \ldots, N)$ with string parameter s. In terms of the viewpoint of topological defect, the vector function $\vec{\phi}$ is just the orderparameter of the defects and these singular strings are just the topological defects.

In δ-function theory[15], one can prove that in three dimension space

$$\delta^2(\vec{\phi}) = \sum_{k=1}^{N}\beta_k\int_{L_k}\frac{\delta^3(\vec{x} - \vec{x}_k(s))}{\mid D(\frac{\phi}{u})\mid_{\Sigma_k}}ds, \tag{33}$$

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where $D(\frac{\phi}{u}) = \frac{1}{2}\epsilon^{\mu\nu}\epsilon_{ab}\frac{\partial\phi^a}{\partial u^\mu}\frac{\partial\phi^b}{\partial u^\nu}$ and Σ_k is the kth planar element transverse to L_k with local coordinates (u^1, u^2). The positive integer β_k is the Hopf index of ϕ-mapping, which means that when $\vec{x}$ covers the neighborhood of the zero point $\vec{x}_k(s)$ once, the vector field $\vec{\phi}$ covers the corresponding region in ϕ space β_k times. Meanwhile from Eq.(33), one have

$$\partial_\mu\phi^a dx^\mu|_{L_k} = 0, \tag{34}$$

then the tangent vector of L_k is given by

$$\frac{dx^\lambda}{ds}\Big|_{L_k} = \frac{D^\lambda(\frac{\phi}{x})}{D(\frac{\phi}{u})}\Big|_{L_k}. \tag{35}$$

Then the inner topological structure of j^λ is

$$j^\lambda = \sum_{k=1}^{N} W_k \int_{L_k} \frac{dx^\lambda}{ds}\delta^3(\vec{x} - \vec{z}_k(s))ds, \tag{36}$$

where $W_k = \beta_k\eta_k$ is the winding number of $\vec{\phi}$ around L_k, with $\eta_k = sgnD(\frac{\phi}{u})|_{\Sigma_k} = \pm 1$ being the Brouwer degree of ϕ mapping. The topological charge of the defect line L_k is

$$Q_k = \int_{\Sigma_k} j^\lambda d\sigma_\lambda = W_k. \tag{37}$$

Using Eq.(25) and (38), the part of SU(2) Chern-Simons action that we are care for is expressed as

$$S = \frac{1}{2\pi}\int_V C_\lambda j^\lambda d^3x = \frac{1}{2\pi}\sum_{k=1}^{l} W_k \int_{L_k} C_\lambda dx^\lambda. \tag{38}$$

It can be seen that when these singular strings are closed curves or more generally are a family of N knots $\gamma_k (k = 1, 2, \ldots, N)$, the inner structure of topological current is

$$j^\lambda = \sum_{k=1}^{N} W_k \oint_{\gamma_k} \frac{dx^\lambda}{ds}\delta^3(\vec{x} - \vec{z}_i(s))ds, \tag{39}$$

and SU(2) Chern-Simons action is

$$S = \frac{1}{2\pi}\sum_{k=1}^{l} W_k \oint_{\gamma_k} C_\lambda dx^\lambda. \tag{40}$$

Consider the infinitesimal gauge transformation $\vec{\alpha} = \alpha\vec{n}$, then C_λ transforms like this

$$C'_\lambda = C_\lambda + \partial_\lambda\alpha, \tag{41}$$

which is just the U(1) gauge transformation. It is seen that the $\partial_\lambda \alpha$ term in Eq.(43) contributes nothing to the integral in Eq.(42). Hence the expression (42) is invariant under the infinitesimal gauge transformation along the direction of $\vec{n}$. These closed singular strings are just the knot structure in SU(2) Chern-Simons field theory.

At last, we must point out that, in this section, we have used the regular condition $D^\lambda(\phi/x) \neq 0$. Generally, this condition is not always tenable. When this condition fails, branch process will occur. A further study of the branch process will appeared in our future paper.

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